OPAKOVÁNÍ

A) ROVNICE A NEROVNICE

Příklad 1. Řešte v $\mathbb{R}$:

1) $2x^2 - 3x - 5 = 0$
$D = (-3)^2 - 4 \cdot 2 \cdot (-5) = 9 + 40 = 49$

$$x_{1,2} = \frac{-(-3) \pm \sqrt{49}}{2 \cdot 2} = \frac{3 \pm 7}{4} = \begin{cases} x_1 = \dfrac{3+7}{4} = \dfrac{10}{4} = \underline{\underline{\dfrac{5}{2}}} \\ x_2 = \dfrac{3-7}{4} = \dfrac{-4}{4} = \underline{\underline{-1}} \end{cases}$$

2) Řešení pomocí Viètových vzorců:

$$x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$
$$x - 3 = 0 \quad \vee \quad x + 5 = 0$$
$$\underline{\underline{x_1 = 3}} \qquad \underline{\underline{x_2 = -5}}$$

Řešení pomocí Viètových vzorců: hledáme taková dvě čísla, aby jejich součin byl -15 a jejich součet 2.

3) $x^2 - |x-1| = 1$
Nulový bod výrazu uvnitř absolotní hodnoty je 1.

	$(-\infty, 1\rangle$	$(1, -\infty)$
$x - 1$	$-$	$+$

- $x \in (-\infty, 1\rangle$

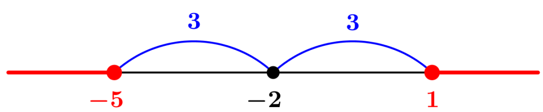
$$x^2 - (-x + 1) = 1$$
$$x^2 + x - 1 = 1$$
$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$\underline{\underline{x_1 = 1}} \in (-\infty, 1\rangle$$
$$\underline{\underline{x_2 = -2}} \in (-\infty, 1\rangle$$

- $x \in (1, -\infty)$

$$x^2 - (x - 1) = 1$$
$$x^2 - x + 1 = 1$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x_3 = 0 \notin (1, -\infty)$$
$$x_4 = 1 \notin (1, -\infty)$$

Řešení: $\underline{\underline{x \in \{-2, 1\}}}$

4) $|x + 2| \geq 3$
$|x - (-2)| \geq 3$

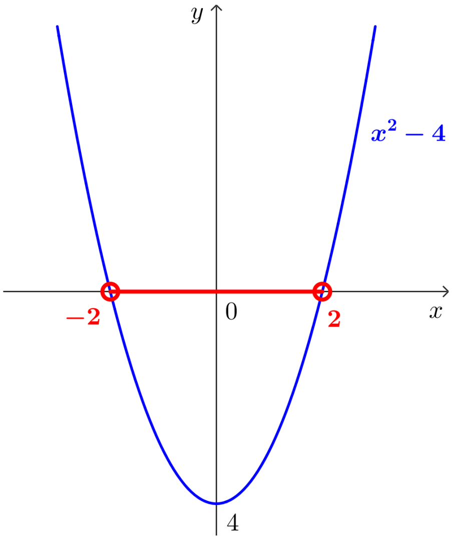


Řešení: $\underline{\underline{x \in (-\infty, -5\rangle \cup \langle 1, \infty)}}$

5) $x^2 < 4$

I. způsob:

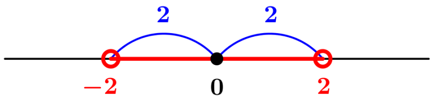
$$x^2 < 4 \quad \rightarrow \quad x^2 = 4$$
$$x^2 - 4 < 0 \qquad x = \pm 2$$
$$\downarrow$$



$x \in (-2, 2)$

II. způsob:

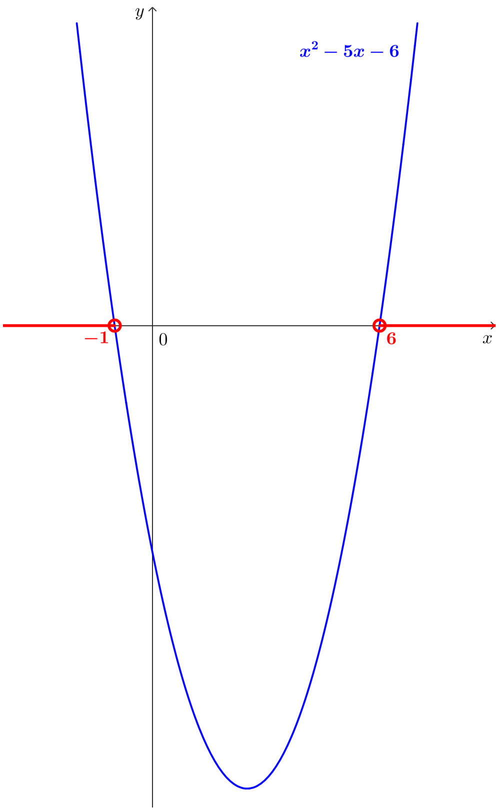
$$x^2 < 4$$
$$|x| < 2$$
$$|x - 0| < 2$$



6) $x^2 - 5x - 6 > 0 \qquad \rightarrow$

$$\downarrow$$

$$x^2 - 5x - 6 = 0$$
$$(x+1)(x-6) = 0$$
$$x_1 = -1 \qquad x_2 = 6$$



$x \in (-\infty, -1) \cup (6, \infty)$

7) $4^{x+1} + 8 \cdot 4^x = 12$

$$4 \cdot 4^x + 8 \cdot 4^x = 12$$
$$12 \cdot 4^x = 12$$
$$4^x = 1$$
$$4^x = 4^0$$
$$\underline{\underline{x = 0}}$$

8) $25^{-x} + 5^{1-x} = 50$

$$(5^{-x})^2 + 5 \cdot 5^{-x} - 50 = 0 \qquad \ldots \qquad \text{substituce: } t = 5^{-x}$$
$$t^2 + 5t - 50 = 0$$
$$(t-5)(t+10) = 0$$
$$t_1 = 5 \quad \rightarrow \quad 5^{-x} = 5$$
$$-x = 1$$
$$\underline{\underline{x = -1}}$$
$$t_2 = -10 \quad \rightarrow \quad \text{nemá řešení, protože } 5^{-x} > 0 \ \forall x \in \mathbb{R}$$

9) $\log_3{(x+1)} + \log_3{(x+3)} = 1$

Podmínky řešitelnosti:

$$x + 1 > 0 \quad \wedge \quad x + 3 > 0$$
$$x > -1 \qquad\qquad x > -3$$
$$x \in (-1, \infty)$$

$$\log_3{[(x+1)(x+3)]} = \log_3 3$$
$$(x+1)(x+3) = 3$$
$$x^2 + 4x + 3 = 3$$
$$x^2 + 4x = 0$$
$$x(x+4) = 0$$
$$\underline{\underline{x_1 = 0}} \in (-1, \infty)$$
$$x_2 = -4 \notin (-1, \infty)$$

10) $2^{\ln x} - 4^{\ln x} = 0 \qquad \ldots \qquad$ Podmínky řešitelnosti: $x \in \mathbb{R}^+$

$$2^{\ln x} - (2^{\ln x})^2 = 0 \qquad \ldots \qquad \text{substituce: } t = 2^{\ln x}$$
$$t - t^2 = 0$$
$$t(1-t) = 0$$
$$t_1 = 0 \quad \rightarrow \quad 2^{\ln x} = 0 \quad \ldots \quad \text{nemá řešení, protože } 2^y > 0 \ \forall y \in \mathbb{R}$$
$$t_2 = 1 \quad \rightarrow \quad 2^{\ln x} = 1$$
$$2^{\ln x} = 2^0$$
$$\ln x = 0$$
$$\underline{\underline{x = 1}} \in \mathbb{R}^+$$

11) $3 \operatorname{cotg} \frac{x}{2} = -\sqrt{3}$

$$\operatorname{cotg} \frac{x}{2} = -\frac{\sqrt{3}}{3} \qquad \Rightarrow \qquad \frac{x}{2} = \frac{2}{3} + k\pi$$
$$\underline{\underline{x = \frac{4}{3} + 2k\pi,\ k \in \mathbb{Z}}}$$

12) $\sin\left(x+\frac{\pi}{6}\right)=\frac{1}{2}$

$$x+\frac{\pi}{6}=\frac{\pi}{6}+2k\pi \qquad \vee \qquad x+\frac{\pi}{6}=\frac{5}{6}\pi+2k\pi$$

$$\underline{\underline{x_1=2k\pi}},\ k\in\mathbb{Z} \qquad\qquad \underline{\underline{x_2=\frac{2}{3}\pi+2k\pi}},\ k\in\mathbb{Z}$$

B) ÚPRAVY VÝRAZŮ

Příklad 2. Upravte výrazy na co nejjednodušší tvar:

1) $$\frac{2x}{x+2}-\frac{6x}{6-3x}+\frac{8x}{x^2-4}=\frac{2x}{x+2}-\frac{6x}{3(2-x)}+\frac{8x}{x^2-4}=\frac{2x}{x+2}+\frac{2x}{x-2}+\frac{8x}{x^2-4}=$$
$$=\frac{2x(x-2)+2x(x+2)+8x}{x^2-4}=\frac{2x^2-4x+2x^2+4x+8x}{x^2-4}=\frac{4x^2+8x}{x^2-4}=$$
$$=\frac{4x(x+2)}{(x+2)(x-2)}=\underline{\underline{\frac{4x}{x-2}}}$$

2) $$\frac{\frac{x}{x-1}-\frac{x+1}{x}}{\frac{x}{x-1}-1}=\frac{\frac{x^2-(x+1)(x-1)}{x(x-1)}}{\frac{x-(x-1)}{x-1}}=\frac{\frac{x^2-(x^2-1)}{x(x-1)}}{\frac{x-x+1}{x-1}}=\frac{x^2-x^2+1}{x}=\underline{\underline{\frac{1}{x}}}$$

3) $$\frac{a^3-b^3}{a+\frac{b^2}{a+b}}=\frac{(a-b)(a^2+ab+b^2)}{\frac{a^2+ab+b^2}{a+b}}=(a-b)(a+b)=\underline{\underline{a^2-b^2}}$$

4) $$\left(\frac{x-1}{x-2}-\frac{x}{x-1}\right)\cdot\left(x-\frac{x}{x+1}\right)\cdot(x^2-1)=$$
$$=\frac{(x-1)^2-x(x-2)}{(x-1)(x-2)}\cdot\frac{x(x+1)-x}{x+1}\cdot(x+1)(x-1)=$$
$$=\frac{(x^2-2x+1-x^2+2x)(x^2+x-x)}{x-2}=\underline{\underline{\frac{x^2}{x-2}}}$$

5) $$\frac{\frac{b}{a^2+ab}+\frac{2}{a+b}+\frac{a}{b^2+ab}}{\frac{a}{b}-\frac{b}{a}}=\frac{\frac{b}{a(a+b)}+\frac{2}{a+b}+\frac{a}{b(b+a)}}{\frac{a}{b}-\frac{b}{a}}=\frac{\frac{b^2+2ab+a^2}{ab(a+b)}}{\frac{a^2-b^2}{ab}}=$$
$$=\frac{\frac{(a+b)^2}{a+b}}{(a+b)(a-b)}=\frac{(a+b)^2}{(a+b)^2(a-b)}=\underline{\underline{\frac{1}{a-b}}}$$

6) $$\frac{\dfrac{1}{x+y}+\dfrac{x-y}{(x+y)^2}}{1+\left(\dfrac{x-y}{x+y}\right)^2}=\frac{\dfrac{x+y+x-y}{(x+y)^2}}{\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2}}=\frac{2x}{x^2+2xy+y^2+x^2-2xy+y^2}=$$
$$=\frac{2x}{2x^2+2y^2}=\frac{2x}{2(x^2+y^2)}=\underline{\underline{\frac{x}{x^2+y^2}}}$$

7) $$\frac{\dfrac{a^2+4}{a}-2}{\left(\dfrac{1}{a^2}+\dfrac{1}{4}\right)\cdot\dfrac{a^3+8}{a^2+4}}=\frac{\dfrac{a^2+4-2a}{a}}{\dfrac{4+a^2}{4a^2}\cdot\dfrac{(a+2)(a^2-2a+4)}{a^2+4}}=\underline{\underline{\frac{4a}{a+2}}}$$

8) $$\frac{\left(ab-\dfrac{1}{ab}\right)^2}{\left(a+\dfrac{1}{b}\right)^2\cdot\left(b-\dfrac{1}{a}\right)^3}=\frac{\left(\dfrac{a^2b^2-1}{ab}\right)^2}{\left(\dfrac{ab+1}{b}\right)^2\cdot\left(\dfrac{ab-1}{a}\right)^3}=\frac{\dfrac{(a^2b^2-1)^2}{a^2b^2}}{\dfrac{(ab+1)^2}{b^2}\cdot\dfrac{(ab-1)^3}{a^3}}=$$
$$=\frac{\dfrac{(ab+1)^2(ab-1)^2}{a^2b^2}}{\dfrac{(ab+1)^2}{b^2}\cdot\dfrac{(ab-1)^3}{a^3}}=\underline{\underline{\frac{a}{ab-1}}}$$

9) $$\frac{(x^2-y^2)^2}{x^3-y^3}\cdot\frac{x^2+xy+y^2}{x^3+y^3}\cdot\frac{1}{x^2+2xy+y^2}+\frac{\dfrac{1-x}{x-x^2}}{x-y+\dfrac{y^2}{x}}=$$
$$=\frac{(x+y)^2(x-y)^2}{(x-y)(x^2+xy+y^2)}\cdot\frac{x^2+xy+y^2}{(x+y)(x^2-xy+y^2)}\cdot\frac{1}{(x+y)^2}+\frac{\dfrac{1-x}{x(1-x)}}{\dfrac{x^2-xy+y^2}{x}}=$$
$$=\frac{x-y}{(x+y)(x^2-xy+y^2)}+\frac{1}{x^2-xy+y^2}=\frac{x-y+x+y}{x^3+y^3}=\underline{\underline{\frac{2x}{x^3+y^3}}}$$

10) $$\frac{\dfrac{a^2+b^2}{b}+2a}{\dfrac{1}{b}+\dfrac{1}{a}}+\frac{2b-\dfrac{a^2+b^2}{a}}{\dfrac{1}{b}-\dfrac{1}{a}}=\frac{\dfrac{a^2+b^2+2ab}{b}}{\dfrac{a+b}{ab}}+\frac{\dfrac{2ab-a^2-b^2}{a}}{\dfrac{a-b}{ab}}=\frac{\dfrac{(a+b)^2}{b}}{\dfrac{a+b}{ab}}+\frac{-\dfrac{(a-b)^2}{a}}{\dfrac{a-b}{ab}}=$$
$$=a(a+b)-b(a-b)=a^2+ab-ba+b^2=\underline{\underline{a^2+b^2}}$$

Příklad 3. Upravte výrazy na co nejjednodušší tvar:

1) $$\frac{\cos^2 x}{1+\sin x} = \frac{1-\sin^2 x}{1+\sin x} = \frac{(1+\sin x)(1-\sin x)}{1+\sin x} = \underline{\underline{1-\sin x}}$$

2) $$\frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x\cos x}{1-(\cos^2 x-\sin^2 x)} = \frac{2\sin x\cos x}{1-\cos^2 x+\sin^2 x} = \frac{2\sin x\cos x}{2\sin^2 x} = $$
$$= \frac{\cos x}{\sin x} = \underline{\underline{\operatorname{cotg} x}}$$

3) $$\frac{1}{1+\operatorname{tg}^2 x} + \frac{1}{1+\operatorname{cotg}^2 x} = \frac{1}{1+\left(\dfrac{\sin x}{\cos x}\right)^2} + \frac{1}{1+\left(\dfrac{\cos x}{\sin x}\right)^2} =$$
$$= \frac{1}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}} + \frac{1}{\dfrac{\sin^2 x+\cos^2 x}{\sin^2 x}} = \frac{1}{\dfrac{1}{\cos^2 x}} + \frac{1}{\dfrac{1}{\sin^2 x}} = \cos^2 x + \sin^2 x = \underline{\underline{1}}$$

4) $$\frac{\sin x}{1+\cos x} + \frac{\sin x}{1-\cos x} = \frac{\sin x(1-\cos x)+\sin x(1+\cos x)}{(1+\cos x)(1-\cos x)} =$$
$$= \frac{\sin x(1-\cos x+1+\cos x)}{1-\cos^2 x} = \frac{2\sin x}{\sin^2 x} = \underline{\underline{\frac{2}{\sin x}}}$$

5) $$\frac{1+\cos 2x}{1-\cos 2x} = \frac{1+\cos^2 x-\sin^2 x}{1-\cos^2 x+\sin^2 x} = \frac{2\cos^2 x}{2\sin^2 x} = \underline{\underline{\operatorname{cotg}^2 x}}$$

6) $$\frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin^2 x}{\cos^2 x}} = \frac{\dfrac{\sin x}{\cos x}}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}} = \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos^2 x}} = \sin x\cos x = \underline{\underline{\frac{1}{2}\sin 2x}}$$

7) $$\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x} = \frac{1-\cos^2 x+\sin^2 x+2\sin x\cos x}{1+\cos^2 x-\sin^2 x+2\sin x\cos x} = \frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x} =$$
$$= \frac{2\sin x(\sin x+\cos x)}{2\cos x(\cos x+\sin x)} = \frac{\sin x}{\cos x} = \underline{\underline{\operatorname{tg} x}}$$

8) $$\frac{\sin x-\sin^3 x}{\cos x-\cos^3 x} = \frac{\sin x(1-\sin^2 x)}{\cos x(1-\cos^2 x)} = \frac{\sin x\cos^2 x}{\cos x\sin^2 x} = \frac{\cos x}{\sin x} = \underline{\underline{\operatorname{cotg} x}}$$

9) $$\frac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=\frac{2\sin x-2\sin x\cos x}{2\sin x+2\sin x\cos x}=\frac{2\sin x(1-\cos x)}{2\sin x(1+\cos x)}=\frac{1-\cos x}{1+\cos x}=\frac{\dfrac{1-\cos x}{2}}{\dfrac{1+\cos x}{2}}=$$

$$=\frac{\sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2}}=\underline{\underline{\operatorname{tg}^2\frac{x}{2}}}$$

10) $$\frac{\operatorname{tg}x\cdot\operatorname{tg}2x}{\operatorname{tg}x-\operatorname{tg}2x}=\frac{\dfrac{\sin x}{\cos x}\cdot\dfrac{\sin 2x}{\cos 2x}}{\dfrac{\sin x}{\cos x}-\dfrac{\sin 2x}{\cos 2x}}=\frac{\dfrac{\sin x}{\cos x}\cdot\dfrac{2\sin x\cos x}{\cos^2 x-\sin^2 x}}{\dfrac{\sin x}{\cos x}-\dfrac{2\sin x\cos x}{\cos^2 x-\sin^2 x}}=$$

$$=\frac{\dfrac{2\sin^2 x\cos x}{\cos x(\cos^2 x-\sin^2 x)}}{\dfrac{\sin x(\cos^2 x-\sin^2 x)-2\sin x\cos^2 x}{\cos x(\cos^2 x-\sin^2 x)}}=\frac{2\sin^2 x\cos x}{\sin x\cos^2 x-\sin^3 x-2\sin x\cos^2 x}=$$

$$=\frac{2\sin^2 x\cos x}{-\sin^3 x-\sin x\cos^2 x}=\frac{2\sin^2 x\cos x}{-\sin x(\sin^2 x+\cos^2 x)}=-2\sin x\cos x=\underline{\underline{-\sin 2x}}$$